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Continuum Aspects of rf Gradient Acceleration of Plasma

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A quasi-one-dimensional model of acceleration of a continuum plasma by rf electric field gradient forces is examined. The conditions imposd on the electric field in order to effect sonic transitions are given. The case of isentropic flow is treated, and departure from the particle model is indicated for pressure >100 μ and temperatures $\simeq$0.1 ev.

Nomenclature

A = cross-sectional area of channel
c = sonic speed
c_p = specific heat at constant pressure
c_v = specific heat at constant volume
D = electric displacement flux
E = electric field intensity
E_a = applied electric field
F_{em} = force/unit volume
K = dielectric constant
L = system constant
l = e^2/mM'
M = sonic Mach number = v/c
M' = ion mass
m = electron mass
p = kinetic pressure
Q = specific heat input
R = universal gas constant
T = temperature
t = time
v = velocity
x = axial coordinate
β = constant in adiabatic gas law
γ = specific heat ratio
ϵ = permittivity of free space
λ = fractional ionization
μ_0 = $\rho_0 v_0$ = initial momentum
ρ = mass density
σ = electrical conductivity
τ = $\beta\mu_0{}^{\gamma-1}[\gamma/(\gamma-1)]$
ω = applied frequency
ω_p = plasma frequency = $(\lambda\rho e^2/\epsilon m M')^{1/2} = (l\rho)^{1/2}$

Acceleration of an electrically neutral plasma by application of a spatially inhomogeneous nonpropagating rf electric field has been demonstrated experimentally[1] at 140, 330, and 2424 Mc. Theoretical studies,[2] using a particle model of the plasma, have predicted the final or exit velocity in terms of the field applied.

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The expression for the time-averaged acceleration on the basis of this model is

$$\ddot{x} = \frac{e^2}{4mM'}\,\frac{1}{\omega^2 - \omega_p{}^2}\,\nabla E_a{}^2 \tag{1}$$

However, in some of the forementioned experiments it would appear appropriate to apply continuum analysis for certain portions of the accelerating regions because of the size of the mean free path involved. In this paper a one-dimensional channel flow is examined in which the forces are time-averaged. The conditions for sonic transition are stated, and for the case of isentropic flow the exit velocity is given in terms of the electric field maximum.

Analysis of the Flow

In the presence of electric and magnetic fields an electrically conductive fluid is subjected to a body force. The expression for such a force density can be obtained by variation of the electromagnetic free energy integral.[3]

Those terms in the expression for the force density which are relevent to the purposes at hand are as follows:

$$F_{em} = \frac{\epsilon}{2}\nabla\left(E^2\rho\,\frac{dK}{d\rho}\right) - \frac{\epsilon}{2}\,E^2\nabla K \tag{2}$$

The specific form of the terms in (2) is, in part, because the variation was performed isothermally. Hence, rigorously speaking, thermal energy changes in the fluid which accompany the field changes are not considered. Therefore, the variation from which the expression for the force density is obtained corresponds only to the variation of the available or electromagnetic "free" energy. Nonetheless, if changes in the value of the dielectric constant which are due to temperature are small, then their effects can be ignored in the expression F_{em}. At the same time, the effects of temperature changes may be included in the energy conservation statement.

The temporal variation in the problem to be considered is avoided by using time-averaged quantities. This treatment is justified by the fact that the ionic constituent of the plasma is relatively motionless during a period corresponding to one cycle. At the same time, however, the field does induce a net electronic displacement and hence gives rise to the polarization of the medium which reacts with the field.

The one-dimensional form of the conservation equations can be written as

$$\rho v A = \text{const} \tag{3}$$

$$\rho v\,\frac{dv}{dx} + \frac{dp}{dx} = \frac{\epsilon}{2}\,(K-1)\,\frac{dE^2}{dx} \tag{4}$$

$$\rho c_p v\,\frac{dT}{dx} + \rho v^2\,\frac{dv}{dx} + \frac{v}{2}\,\frac{d}{dx}\,(E\cdot D) = Q + \sigma E^2 \tag{5}$$

In Eq. (4), the explicit form for the expression F_{em} has been inserted. This is justified on the following basis. K is assumed to be that expression which is derivable on the basis of a linear harmonic analysis, that is, $K = 1 - \alpha\rho$, where $\alpha = l/\omega^2$. This is probably quite good except at frequencies very near the resonance of the plasma. Here it is expected that nonlinear effects become important. Even so, K† must change sign in going from one side of the resonance to the other, and this is the important fact relevent to the transition.

† Physically, K goes positive when the applied frequency goes higher than the plasma frequency, because the phase of the electronic motion falls sufficiently far behind. The result of this is that the polarization of the induced field becomes parallel to that of the applied field.

On these foregoing bases, the reduced equations give rise to the following expression for the velocity change:

$$\frac{dv}{dx} = \frac{1}{M^2 - 1}\left[\frac{v}{A}\frac{dA}{dx} - \frac{\gamma - 1}{\gamma}\frac{Q}{2p} - \sigma\frac{\gamma - 1}{\gamma}\frac{E^2}{p} + \frac{\gamma - 1}{\gamma}\frac{v}{p}\frac{d}{dx}\left(\frac{E\cdot D}{2}\right) + \frac{v}{p}\frac{\epsilon}{2}(K - 1)\frac{dE^2}{dx}\right] \quad (6)$$

The first two terms are familiar from ordinary gasdynamics and partially specify the conditions under which the transition from subsonic to supersonic flow may be obtained. The remaining terms specify the electrical effects that also partially determine transition conditions.

In order to clarify the effects of the electric field on the transition, consider the situation corresponding to a constant area channel with no heat input and with negligible conductivity. In addition, the relation $D = K\epsilon E$ is assumed. In this case, the expression reduces to the following:

$$\frac{dv}{dx} = \frac{1}{M^2 - 1}\left\{\frac{v}{p}\frac{1}{2\epsilon}\frac{D^2}{K^2}\right\}\left\{\frac{1}{D^2}\frac{dD^2}{dx}\left[\left(\frac{2\gamma - 1}{\gamma}\right)K - 1\right] - \frac{1}{K}\frac{dK}{dx}\left[\left(\frac{3\gamma - 1}{\gamma}\right)K - 2\right]\right\} \quad (7)$$

In the interpretation of this expression, it is assumed that the sign of the square of the displacement gradient is unchanged by the flow. That is to say that the flow, although it is instrumental in determining the local value of the displacement, is not such as to cause the gradient of the latter to reverse direction. Also, since $dv/dx > 0$ for positive acceleration, and since $\rho v = \text{const}$, then $d\rho/dx < 0$ for such acceleration. Since it is assumed that $K = 1 - \alpha\rho$, then with $dv/dx > 0$ it follows that $dK/dx > 0$. With the braces in (7) positive, this equation indicates the following:

1) For a flow initially subsonic, the velocity change will be positive when K is less than zero and the displacement gradient is positive.

2) The sonic transition can be effected at $K = 0$ if both the displacement gradient and the gradient of the dielectric constant go to zero at this value of K. A velocity increase in the subsonic regime is possible even above $K = 0$, but this latter circumstance appears somewhat less desirable because the value of K necessary for a sonic transition will then depend upon both the displacement gradient and the dielectric constant gradient.

3) In the supersonic regime with $[0 < K < \gamma/(2\gamma - 1)]$, the acceleration will be positive if the gradient becomes negative. (On the basis of practical considerations, the field gradient must be negative in the terminating section of a device designed to implement this force principle in order that the fluid may see essentially zero field at the exit of the accelerating section.) It is possible to continue the velocity increase in the supersonic regime above $K = \gamma/(2\gamma - 1)$, but a complication similar to the subsonic case arises.

A simple, consistent acceleration scheme can be obtained from the foregoing:

1) For an initial velocity that is subsonic, the dielectric constant should start at values less than zero.

2) The sonic transition can be effected at $K = 0$ with both the displacement gradient and the dielectric constant gradient at zero.

3) The supersonic regime should end with an exit density (density at zero field) such that $K \approx \gamma/(2\gamma - 1)$.

4) For an initially supersonic beam, the initial value of the dielectric constant should be greater than zero, and, as previously mentioned, the exit density should be such that $K \approx \gamma/(2\gamma - 1)$.

In effect, in order to insure a transition to supersonic flow, the position at which the plasma is generated relative to the field maximum is determined by the initial conditions on the plasma.

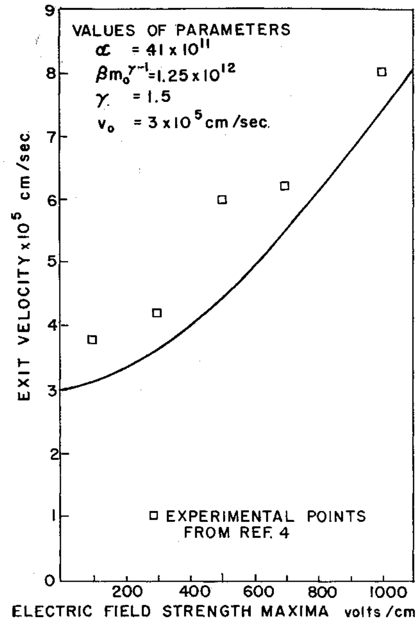

Fig. 1 Comparison between theory and experiment-exit velocity vs maximum field strength.

By specializing to isentropic flow, Eqs. (3–5) are amended to permit easy integration. In this case the adiabatic gas law holds (i.e., $p = \beta\rho\gamma$). The integration yields a constant of the motion which gives the expression for the velocity in terms of the electric field. This expression is

$$(v^2/2) - \tau v^{1-\gamma} + (\epsilon/2)\alpha E^2 = L \quad (8)$$

The expression differs from the expression obtained on the basis of a particle approach in that the second term on the left includes the effects of pressure and temperature changes.

In order to compare experiment with theory, the experimental arrangements for the work at 2.4 kMc are discussed here briefly for the purpose of determining the value of the constants in (8). In this experiment, a pulse of plasma was generated by discharging a condenser through a pool of mercury. This plasma issued out of a small hole and into a microwave cavity fed beyond cutoff. The pulse of plasma which was generated in the device lasted a little over 100 μsec, and approximately 0.08 j were dissipated during this time. On the basis of 73 cal/g for the heat of vaporization of mercury, it is calculated that somewhat in excess of 10^{16} atoms of mercury were released during a pulse. About midway through the plasma pulse, the rf was pulsed on for 10 μsec. (The velocity without rf field was measured to be at 3×10^5 cm/sec.) Although the background density of mercury corresponded to about $10^{14}/\text{cm}^3$, the density during the major fraction of the plasma pulse, and hence during the time of application of the electric field inside and at the face of the stub, was greater than $10^{15}/\text{cm}^3$. Thus, an initial density of $5 \times 10^{15}/\text{cm}^3$ was assumed for evaluation of the constants. It is interesting to note that such a density is required in order for equilibrium to be established between the rate of generation of mercury atoms and their efflux rate out of the stub. If, as assumed previously, the adiabatic gas law holds during the time the plasma is being pulsed, then the required scaling up of the constants in (8) from their value prior to pulsing can be made.

The plot of the theoretical curve for Eq. (8) given in Fig. 1 is for values of the internal field, as distinguished from the other data[4] in the figure which are given in terms of the external fields. The final axial velocity is plotted against the electric field maximum. It is preferable to express the results in terms of the external field. However, in general, determining the relation between the external and internal fields in a problem involving the flow of electrically conducting fluid implies the solutions of a coupled electromagnetic field-velocity field problem. Moreover, for very dense plasmas, it is implied that viscosity effects must be included because of their effect on the velocity field near the boundaries of the flow. Nonetheless, it is believed that the magnitudes of each are not significantly different in certain cases. For the example discussed herein, the relatively good agreement between the experimental data and the theory suggests that this is the case.

The results of the continuum theory as applied to the experimental situation do not differ in a significant way from the results of the particle approach. An essentially parabolic curve is obtained. This is primarily because of the relative predominance of the initial velocity. This velocity is an experimentally measured value. The experimental arrangement suggests, however, that this measured velocity is more representative of a portion of the ionized fraction of the plasma than of the plasma as a whole. If the velocity of the neutrals were properly weighted (this is assumed in the continuum theory), the initial velocity of the plasma would likely be lower.

At higher densities, the continuum equations represent the motion more realistically. The second term in Eq. (8) which takes into account the effects of collisions through such variables as temperature and pressure will exercise greater importance as the density increases, and hence the departure from the parabolic dependence of exit velocity on field maximum would become more pronounced. Lower initial velocities would also cause a departure from the parabolic curve.

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Three-Dimensional Supersonic Flow Computations

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THE numerical solution of systems of partial differential equations with more than two independent variables was made possible in recent years by large-scale, high-speed digital computers. However, no complete numerical analysis of three-dimensional supersonic flow fields appeared in the literature until 1961, when some of the results summarized in this note were obtained at General Applied Science Laboratories Inc.

Three-dimensional computations are difficult to perform. In building up mathematical arguments, the number of dimensions is irrelevant, but practical computations need the help of geometry, which is limited by the two-dimensionality of the paper and the blackboard. An extensive use of two-dimensional geometry allowed the author and his collaborators to achieve large-scale results in a surprisingly short time without spoiling the three-dimensional nature of the problems.

Let u, v, and w be the components of the velocity $\bar{V}$ with respect to three orthogonal axes x_1, x_2, and x_3, in order. Move to the right-hand side of the equations of motion all terms that do not contain derivatives and the terms that contain derivatives with respect to x_2. Hence, the left-hand sides of the continuity equation and of the first and third scaler momentum equations contain only ρ, p, u, and w and their derivatives with respect to x_1 and x_3. If the right-hand sides are considered as known, this system of three equations depends on x_1 and x_3 only, and its characteristics can be determined by a standard procedure. They are formally the same as in a two-dimensional problem. The compatibility equations along the characteristics can be written in a similar way, but with additional terms that involve v and the derivatives with respect to x_2. This idea, stemming on early theories,[1] is outlined in Ref. 2.

Any of the three compatibility equations and the second momentum equation can be written in the form

$$U_k + BV_k = F$$

where U and V in turn stand for $\tau(= u/w)$, $q^2(= u^2 + w^2)$, v, and p. The subscript k means differentiation along a characteristic ($k = 1,2,3$); B is a function of u, w, p, and ρ; and F is a function of these variables and of their derivatives with respect to x_2. These equations have been derived from the original equations of motion without linearization or simplifications of any kind. They suggest a finite difference technique that can be described as follows:

1) Consider one $x_3 = x_{30}$ surface and several $x_2 = x_{2i}$ lines on it. Let A_{ik} be a point on one of these lines.

2) Compute the derivatives with respect to x_2 and the other parameters in B and F at each point A_{ik}.

3) Step off the $x_3 = x_{30}$ surface along characteristics in the (x_1, x_3) surfaces, assuming B and F equal to their initial values through the first intersection of two characteristics.

The difference between the present problem and two-dimensional or axisymmetrical problems, where the assumption in step 3 is a matter of routine, consists in that here not only the parameters but also their cross derivatives must be almost constant throughout the step. A proper choice of the frame of reference is necessary. A sizeable change in some of the cross-derivatives can only occur as a consequence of an abrupt crosswise spreading of the streamlines. If the frame of reference is chosen with its curvilinear axis normal to the general direction of the flow, at each step the v component of the velocity will be small and the cross-derivatives nearly constant.

Now, the general behavior of the flow in a shock layer is a consequence of the body geometry. In the vicinity of a spherical nose, the flow is about axisymmetrical. Along a flat surface it tends to become two-dimensional. Along a rounded leading edge it tends to become cylindrical, and so forth. Consequently, different regions must be analyzed within different frames of reference.

In more sophisticated analyses dealing with three-dimensional characteristic surfaces, this problem does not arise. However, the program becomes substantially simpler if a simple, two-dimensional scheme is used throughout a region. This means that different simple frames of reference must be chosen over different regions of the body in connection with its geometry. The choice can easily be made before starting a computation. All the results have been obtained so far using a Cartesian frame and a cylindrical frame whose first meridional plane is the last Cartesian plane. The spacing between Cartesian planes, the angle between meridional planes, and the origin of the cylindrical frame can be auto-

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